

NONCOMMUTATIVE STABLE HOMOTOPY THEORY AND KK -THEORY

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April 29, 2013

Abstract: We construct Kasparov's bifunctor KK and E -theory by stable homotopy theoretic methods. This is motivated by constructions of bivariant theories on more general categories such as, for example, bornological algebras. The details of the construction have interesting applications to groups of extensions.

In [Ros82], Rosenberg proposed to study monovariant K -theory from a stable homotopy point of view, and formulates the natural conjecture that Brown-Douglas-Fillmore Ext -theory and Kasparov's KK -theory should to some extent be describable in the stable homotopic framework. We will give a positive answer to this question, following ideas of Higson used to define E -theory ([Hig90]); in particular, our constructions simplify the construction of E -theory by separating the stabilisation by compact operators from the stable-homotopy construction.

On the other hand, the bivariant K -functor of Kasparov ([Kas80]) is well known to be characterized by certain exactness, stability and homotopy invariance properties. Cuntz has developed several versions of bivariant K -theories ([Cun05],[Cun97],[CT06]) on the category of (complete) locally convex algebras, that posses desirable properties, referred to as ckk in the sequel. An ultimate generalization has been given by in [CMR07] to bornological algebras, based on the formalism of triangulated categories.

On the other hand, in [Gre12], we have shown, by a direct calculation using what we call locally convex Kasparov modules, that Bott periodicity holds for a larger class than ckk is known to be universal for, namely for arbitrary split exact functors which are diffeotopy invariant and stable. The proof adapts readily to the C^* -setting.

This makes it desirable to develop universal split exact functors using only the Bott periodicity theorem. We here undertake this task for the category of C^* -algebras in a way that adapts readily to the categories of Banach, locally convex and Bornological algebras. At the same time, this extends KK to not necessarily separable C^* -algebras.

We refer to [Gre12] for techniques allowing to calculate explicitly products of elements associated to Kasparov products in the category we construct here abstractly.

1 Stable Homotopy Theory and Exactness

We denote by \mathcal{H} the homotopy category of C^* -algebras, i.e. the category with morphisms $\mathcal{H}(A, B) := C^*(A, B)/\sim$, where \sim denotes homotopy. The definitions of cones, suspensions, cylinders, mapping cones and mapping cylinders are those from [Bla98].

Definition 1. Let \mathcal{SH} be the category with objects A_m , where A is a C^* -algebras and $m \in \mathbb{Z}$. We define the morphisms by

$$\mathcal{SH}(A_m, B_n) := \lim_{k \rightarrow \infty} \mathcal{H}(\mathcal{S}^{m+k} A, \mathcal{S}^{n+k} B)$$

where the connecting maps are given by the functor \mathcal{S} and the limit is taken over all $k \in \mathbb{Z}$ such that $m+k, n+k \geq 0$.

Σ denotes the functor defined by $\Sigma(A_m) := A_{m+1}$.

We will frequently abbreviate A_0 to A . Note that as in the classical setting, the sets $\mathcal{SH}(A_m, B_n)$ carry natural group structures induced by concatenation. We denote by X the class generated by X^0 and all isomorphisms in \mathcal{SH} .

We denote by X^0 the collection of morphisms in of C^* -algebras that are inclusions of kernels of split surjections onto contractible algebras.

Recall that Higson has given a (weak) set of axioms for a calculus of fractions in [Hig90]. We refer the reader to loc. cit. for the exact conditions on a class of morphisms in order to admit a calculus of fractions, and call such a set admissible. It is easy to see that Higson's axioms apply in the more general setting of categories which are not necessarily small, i.e., such that the isomorphism classes of objects form a set. In fact, it suffices to use (for example) Grothendieck's framework of universes to carry over the theory from [Hig90].

The fact that X^0 is an admissible set of morphisms follows essentially by the following two lemmas of independent interest:

Lemma 2. *Let*

$$0 \longrightarrow I \xrightarrow{\iota} A \xrightarrow{\pi} B \longrightarrow 0$$

be a split extension of C^ -algebras and $\varphi : \mathcal{I} \rightarrow \mathcal{I}'$ a morphism. Then, up to homotopy equivalence, there is a diagram with exact rows*

$$\begin{array}{ccccccc} 0 & \longrightarrow & SI & \xrightarrow{S\iota} & SA & \xrightarrow{S\pi} & SB \longrightarrow 0 \\ & & \downarrow S\varphi & & \downarrow & & \parallel \\ 0 & \longrightarrow & SI' & \longrightarrow & A' & \xrightarrow{\pi'} & SB \longrightarrow 0. \end{array}$$

and where the bottom extension is split.

Proof. The Lemma follows by replacing SI' by the homotopy equivalent algebra which is the mapping cone of

$$ev_0 : CI \rightarrow I', f \mapsto \varphi(f(0)).$$

□

Lemma 3. *Let*

$$0 \longrightarrow I' \xrightarrow{\iota'} A' \xrightarrow{\pi'} B' \longrightarrow 0$$

be an extension of C^ -algebras with B' contractible and π' split. Let $\varphi : A \rightarrow A'$ be a morphism. Then up to homotopy equivalence there is a diagram with split exact rows*

$$\begin{array}{ccccccc} 0 & \longrightarrow & I & \xrightarrow{\iota} & A & \xrightarrow{\pi} & B' \longrightarrow 0 \\ & & \downarrow & & \downarrow \varphi & & \parallel \\ 0 & \longrightarrow & I' & \xrightarrow{\iota'} & A' & \xrightarrow{\pi'} & B' \longrightarrow 0 \end{array}$$

Proof. This follows by replacing A with Z_φ

□

Definition 4. We denote by $\mathcal{SH}(X^{-1})$ the localisation of \mathcal{SH} with respect to X .

Theorem 5. For every fixed C^* -algebra D the functors $\mathcal{SH}(X^{-1})(D, \cdot)$ and $\mathcal{SH}(X^{-1})(\cdot, D)$ are split exact. Every split exact, homotopy invariant Bott periodic functor factors uniquely over $\mathcal{SH}(X^{-1})$.

Proof. It follows by standard arguments that for every morphism $\varphi : A \rightarrow B$ one has exact sequences

$$\mathcal{SH}(D, C_\varphi) \xrightarrow{\pi_*} \mathcal{SH}(D, A) \xrightarrow{\varphi_*} \mathcal{SH}(D, B)$$

where $\pi : C_\varphi \rightarrow A$ is the canonical projection from the mapping cone sequence of φ (compare [CS86] or [Ros82]). The image of this extension in $\mathcal{SH}(X^{-1})$ remains exact, as follows from results in [Hig90]. If φ is a split surjection, then one may deduce split exactness by replacing C_φ with $\text{Ker}(\varphi)$. □

All that remains is to stabilize the functor we have thus obtained. We denote by \mathbb{K} the compact operators on a separable Hilbert space.

Theorem 6. *Define*

$$\tilde{K}K(A, B) := \mathcal{SH}(X^{-1})(A \otimes \mathbb{K}, B \otimes \mathbb{K}).$$

Then $\tilde{K}K$ is naturally isomorphic to Kasparov's bivariant K -functor.

Proof. $\tilde{K}K$ is stable because the inclusion $\mathbb{K} \rightarrow \mathbb{K} \otimes \mathbb{K}$ obtained from a minimal projection is invertible up to homotopy. The usual characterization of KK as a universal functor shows that we re-obtain KK . □

Using the same approach, but for the class Y generated by all morphisms of C^* -algebras that are inclusions of kernels of surjections (not necessarily with a split) onto contractable algebras and all isomorphisms in \mathcal{SH} , one obtains E -theory:

Theorem 7. *Set*

$$\tilde{E}(A, B) := \mathcal{SH}(Y^{-1})(A \otimes \mathbb{K}, B \otimes \mathbb{K}).$$

Then \tilde{E} is naturally isomorphic to the functor E defined in [Hig90] and [CH90].

It is also easy to construct intermediate theories as in [Cun98]-our constructions replace the rather involved construction of the "noncommutative suspension J " by the straightforward suspension.

Because $\tilde{K}K$ admits outer products one may now adopt the formal proof from [Gre] to give a proof of the Thom isomorphism based only on Bott periodicity and without reference to the concrete description of KK :

Proposition 8. *Let H be a split exact, homotopy invariant and \mathbb{K} -stable functor. Then*

$$H(C_0(E)) \approx H(C_0(X))$$

for every complex bundle $E \rightarrow X$.

2 Stable homotopy theory of extensions

The Lemma 2 further allows to define a theory of extensions based only on stable homotopy theory without stabilising by matrices. In fact, for any two extensions

$$0 \longrightarrow A \longrightarrow D_i \longrightarrow B \longrightarrow 0$$

of B over A , we may define a sum by first passing to the pull-back along the diagonal $B \rightarrow B \oplus B$, $b \mapsto (b, b)$ to obtain

$$0 \longrightarrow A \oplus A \longrightarrow D_1 \oplus D_2 \longrightarrow B \oplus B \longrightarrow 0$$

Taking the stable homotopy pushout as in Lemma 2 of the extension

$$0 \longrightarrow S(A \oplus A) \longrightarrow S((D_1 \oplus D_2) \oplus_{\Delta} B) \longrightarrow SB \longrightarrow 0 .$$

along the map $SA \oplus SA \rightarrow SA$ given by concatenation then defines a stable homotopy version of the "Baer sum" of extensions.

3 Bivariant K -theory for more general algebras

The constructions outlined above carry over to other categories of algebras such as bornological, locally convex or Banach algebras. In these settings, one uses rather diffeotopy and projective tensor products, together with appropriate versions of cylinders, cones and suspensions.

The only point which differs significantly is the stabilisation. In the more general settings, following Cuntz, one may stabilize by smooth compact operators or the operator ideals \mathcal{L}^p of p -summable operators.

However, it is more difficult to show that the functor obtained as in Theorem 6 is stable. One may instead "formally stabilize" in order to obtain a universal functor. If the inclusion $\mathcal{L}^p \rightarrow \mathcal{L}^p \otimes_{\pi} \mathcal{L}^p$ is invertible up to diffeotopy (a fact which seems very likely to be true, but which we have not been able to prove at present), then the two ways to stabilize can be seen to coincide.

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References

- [Bla98] B. Blackadar. *K -theory for operator algebras*, volume 5 of *Mathematical Sciences Research Institute Publications*. Cambridge University Press, Cambridge, second edition, 1998.
- [CH90] A. Connes and N. Higson. Déformations, morphismes asymptotiques et K -théorie bivariante. *C. R. Acad. Sci. Paris Sér. I Math.*, 311(2):101–106, 1990.
- [CMR07] J. Cuntz, R. Meyer, and J. M. Rosenberg. *Topological and bivariant K -theory*, volume 36 of *Oberwolfach Seminars*. Birkhäuser Verlag, Basel, 2007.

- [CS86] J. Cuntz and G. Skandalis. Mapping cones and exact sequences in KK -theory. J. Operator Theory, 15(1):163–180, 1986.
- [CT06] J. Cuntz and A. Thom. Algebraic K -theory and locally convex algebras. Math. Ann., 334(2):339–371, 2006.
- [Cun97] J. Cuntz. Bivariante K -Theorie für lokalkonvexe Algebren und der Chern-Connes-Charakter. Doc. Math., 2:139–182 (electronic), 1997.
- [Cun98] J. Cuntz. A general construction of bivariant K -theories on the category of C^* -algebras. In Operator algebras and operator theory (Shanghai, 1997), volume 228 of Contemp. Math., pages 31–43. Amer. Math. Soc., Providence, RI, 1998.
- [Cun05] J. Cuntz. Bivariant K -theory and the Weyl algebra. K-Theory, 35(1-2):93–137, 2005.
- [Gre] M. Gensing. The Thom isomorphism in bivariant K -theory. submitted.
- [Gre12] M. Gensing. Universal cycles and homological invariants of locally convex algebras. J. Funct. Anal., 263(8):2170–2204, 2012.
- [Hig90] N. Higson. Categories of fractions and excision in KK -theory. J. Pure Appl. Algebra, 65(2):119–138, 1990.
- [Kas80] G. G. Kasparov. The operator K -functor and extensions of C^* -algebras. Izv. Akad. Nauk SSSR Ser. Mat., 44(3):571–636, 719, 1980.
- [Ros82] Jonathan Rosenberg. The role of K -theory in noncommutative algebraic topology. In Operator algebras and K -theory (San Francisco, Calif., 1981), volume 10 of Contemp. Math., pages 155–182. Amer. Math. Soc., Providence, R.I., 1982.